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Hannay angle and geometric phase shifts under adiabatic parameter changes in classical dissipative systems

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Abstract
Kepler and Kagan (1991 *Phys. Rev. Lett.* **66** 847) derived a geometric phase shift in dissipative limit cycle evolution. This effect was considered as an extension of the geometric phase in classical mechanics. We show that the opposite is also true, namely, this geometric phase can be identified with the classical mechanical Hannay angle in an extended phase space. Our results suggest that this phase can be generalized to a stochastic evolution with an additional noise term in evolution equations.

PACS numbers: 03.65.Vf, 05.10.Gg, 05.40.Ca

The Berry phase in quantum mechanics [1] appeared as a unifying concept. It provided similar mathematical background to seemingly very different quantum mechanical phenomena. After its discovery, various geometric phases were found even beyond quantum mechanics, for example, in classical mechanics [2], hydrodynamics [3], dissipative kinetics [4, 5] and stochastic processes [6–9]. It was possible to relate some of these phases to each other. For example, the Hannay angle can be derived in the classical limit of the quantum mechanical Berry phase [10]. Relations and hierarchy among other geometric phases remain not well understood. In this communication we partly fill this gap, and demonstrate the relation of the geometric phase in dissipative limit cycle evolution to the classical mechanical Hannay angle. Our approach is similar to the one employed in [7] to introduce geometric phases in stochastic processes, which suggests that further connections among various geometric phases can be found.

In [4], Kagan *et al* considered a dissipative system that evolves to a limit cycle so that after fast relaxation processes the only one angle degree of freedom $\phi(t)$ is relevant and evolves according to

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \Omega(\phi, \mu), \tag{1}$$

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where μ is the vector of slowly time-dependent parameters and Ω is the instantaneous rotation frequency. Kagan *et al* introduced another angle variable

$$\theta(\phi, \mu) = \int_0^\phi \frac{\omega(\mu)}{\Omega(\phi', \mu)} \, d\phi', \tag{2}$$

where

$$\omega(\mu) = \left(\int_0^{2\pi} \frac{1}{\Omega(\phi, \mu)} \frac{d\phi}{2\pi} \right)^{-1}, \tag{3}$$

and showed that under the adiabatic cyclic evolution of μ during time T the phase (2) becomes the sum of dynamic and geometric parts, i.e.

$$\theta = \theta_{\text{dyn}} + \theta_{\text{geom}}, \tag{4}$$

where

$$\theta_{\text{dyn}} = \int_0^T dt\, \omega(\mu(t)), \tag{5}$$

and

$$\theta_{\text{geom}} = \oint \mathbf{A} \cdot d\mu, \qquad \mathbf{A} = \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{\omega(\mu(t))}{\Omega(\phi, \mu)} \partial_\mu \theta(\phi, \mu). \tag{6}$$

The authors of [4] argued that the Hannay angle in classical mechanics is merely a special case of this phase. Below we show that in some sense the opposite is also true, namely that the geometric phase (6) follows from canonical equations of motion and is identified with the Hannay angle [10]. Let us introduce the variable Λ, which we assume to be canonically conjugated to ϕ with the Hamiltonian

$$H(\Lambda, \phi) = \Lambda \Omega(\phi, \mu). \tag{7}$$

The phase evolution (1) then follows from equation

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial \Lambda}. \tag{8}$$

In the case of time-independent μ, its conjugated equation

$$\frac{d\Lambda}{dt} = -\frac{\partial H}{\partial \phi} \tag{9}$$

has a solution

$$\Lambda = \frac{E(\mu)}{\Omega(\phi, \mu)}, \tag{10}$$

where E is the energy. The adiabatically conserved quantity is the action defined by

$$I = \frac{1}{2\pi} \int_0^{2\pi} \Lambda(\phi, \mu) \, d\phi = \frac{1}{2\pi} \int_0^{2\pi} \frac{E(\mu)}{\Omega(\phi, \mu)} \, d\phi, \tag{11}$$

from which follows that

$$E(\mu) = I \omega(\mu), \tag{12}$$

and

$$\Lambda = \Lambda(I, \phi) = I \frac{\omega(\mu)}{\Omega(\phi, \mu)}. \tag{13}$$

Expression for the canonically conjugated to I angle variable θ reads

$$\theta = \frac{\partial}{\partial I} \left(\int_0^\phi \Lambda(I, \phi') \, d\phi' \right) = \int_0^\phi \frac{\omega(\mu)}{\Omega(\phi', \mu)} \, d\phi'. \tag{14}$$

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Comparing (2) and (14) we find that the angle variable θ introduced in [4] is just a canonical angle variable in the model with the Hamiltonian $H(\Lambda, \phi)$. This, in fact, justifies the choice of variables made in [4]. After adiabatic evolution in the parameter space, the angle variable becomes a sum of the dynamic part and the Hannay angle [10]:

$$\theta = \theta_{\mathrm{dyn}} + \theta_H, \tag{15}$$

where

$$\theta_{\mathrm{dyn}} = \frac{\partial}{\partial I} \int_0^T \mathrm{d}t\, E(\mu(t)) = \int_0^T \omega(\mu(t))\, \mathrm{d}t, \tag{16}$$

$$\theta_H = -\frac{\partial}{\partial I} \oint \mathrm{d}\mu \langle \Lambda\,(I, \phi(\theta, \mu))\, \partial_\mu \phi(\theta, \mu) \rangle_\theta = \oint \mathbf{A} \cdot \mathrm{d}\mu, \tag{17}$$

and the averaging is over one fast cycle of θ angle. The connection $\mathbf{A}$ explicitly reads

$$\mathbf{A} = -\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \frac{\omega(\mu)}{\Omega(\phi(\theta, \mu), \mu)} \partial_\mu \phi(\theta, \mu). \tag{18}$$

The last step is to show that connections in (18) and (6) are the same. For this note that

$$\frac{\mathrm{d}\phi(\theta, \mu)}{\mathrm{d}t} = \Omega = \frac{\partial \phi}{\partial \theta} \frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\partial \phi}{\partial \mu} \frac{\mathrm{d}\mu}{\mathrm{d}t}, \tag{19}$$

and that

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega(\mu) + \frac{\partial \theta}{\partial \mu} \frac{\mathrm{d}\mu}{\mathrm{d}t}, \qquad \frac{\partial \phi}{\partial \theta} = \frac{\Omega}{\omega}, \tag{20}$$

which lead to

$$\partial_\mu \phi = -\frac{\partial \phi}{\partial \theta} \partial_\mu \theta. \tag{21}$$

Substituting this into (18) and switching to integration over ϕ one recovers equation (6).

In conclusion, we established a relation between the classical mechanical Hannay angle and the geometric phase in dissipative limit cycle evolution. The Hannay angle interpretation of the phase in [4] relates it also to geometric phases in stochastic kinetics, introduced in [7] by similar variable doubling technique, which may be practically interesting. The doubling of variables can be used to promote not only dissipative but also stochastic equations to the Hamiltonian evolution [11]. Thus, it is possible to derive Hamiltonian formulation for equation (1) with an additional noise term. However, the physical meaning of the resulting Hannay angle remains an open problem.

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